

Derivative Formula, Integration by Parts Formula and Applications for SDEs Driven by Fractional Brownian Motion

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Abstract. In the paper, the Bismut derivative formula is established for multidimensional SDEs driven by additive fractional noise ($\frac{1}{2} < H < 1$), and moreover the Harnack inequality is given. Through a Lamperti transform, we will show that the Harnack inequality also holds for one-dimensional SDEs with multiplicative fractional noise. As applications, the strong Feller property is derived and an invariant probability measure is founded for a discrete semigroup. Finally, we will establish the Driver integration by parts formulas for SDEs driven by additive fractional noise ($0 < H < 1$) and present the shifted Harnack inequalities. As a consequence, the law of solution has a density with respect to the Lebesgue measure.

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1 Introduction

The Bismut derivative formula [6] and the Driver integration by parts formula [11] are two important tools in stochastic analysis. Let ∇ be the gradient operator and P_t stand for the diffusion semigroup. Using the two formulas, one can estimate the commutator $\nabla P_t - P_t \nabla$, which plays a key role in the study of flow properties [14]. Based on martingale method, coupling argument or Malliavin calculus, the Bismut derivative formula has been widely studied and applied in various fields, for instance, see [12, 33, 36] and references therein. Whereas, by using a new coupling argument, [34] established general results on integration by parts formula and applied them to various models including degenerate diffusion process, delayed SDEs and semi-linear SPDEs.

On the other hand, dimensional-free Harnack inequality, initiated in [29], has various applications, see, for instance, [7, 25, 26, 30, 31] for strong Feller property and contractivity properties; [1, 2] for short times behaviors of infinite dimensional diffusions; [7, 15] for heat kernel estimates and entropy-cost inequalities. In general, one can establish this type of Harnack inequality by

using the method of derivative formula (e.g. [1, 17, 25, 29]) or the approach of coupling and Girsanov transformations (e.g. [4, 13, 24, 32]).

In this article, we are concerned with the stochastic differential equations driven by fractional Brownian motion. By using the theory of rough path analysis introduced in [18], Coutin and Qian [8] proved an existence and uniqueness result with Hurst parameter $H \in (\frac{1}{4}, \frac{1}{2})$. Based on a fractional integration by parts formula [35], Nualart and Răşcanu [21] established the existence and uniqueness result with $H > \frac{1}{2}$. For the regularity results about the law of the solution, one can see [16, 19, 23] and references therein. Recently, Fan [13] established the Bismut derivative formula and Harnack inequality for SDEs driven by fractional Brownian motion with $H < \frac{1}{2}$, and Saussereau [28] proved transportation inequalities for the law of the solution with $H > \frac{1}{2}$.

The main purpose of our work is to study the Bismut derivative formula and the Driver integration by parts formula for SDEs driven by fractional Brownian motion. First, using coupling argument and the Girsanov transform for fractional Brownian motion, we will show that, in the case of $H > \frac{1}{2}$, the Bismut derivative formula holds for multidimensional equation when the diffusion coefficient is constant. Based on the derivative formula, the Harnack inequality will be given for multidimensional equation with additive fractional noise, and for one-dimensional equation with multiplicative fractional noise. As applications of the Harnack inequality, the strong Feller property is derived and an invariant probability measure is founded for a discrete semigroup. Finally, we will establish the Driver integration by parts formulas for SDEs driven by fractional Brownian motion with $H \in (0, 1)$. Consequently, the shifted Harnack inequalities and the existence of the density with respect to the Lebesgue measure for solution are presented.

The paper is organized as follows. In section 2, we give some preliminaries on fractional Brownian motion. In section 3, we will prove the Bismut derivative formula for multidimensional SDEs with additive fractional noise ($H > \frac{1}{2}$), and moreover obtain the Harnack inequality. Through a Lamperti transform, we will show that the Harnack inequality also holds for one-dimensional SDEs with multiplicative fractional noise. The remains of the part is devoted to establishing the strong Feller property and invariant probability measure. Finally, section 4 discusses the Driver integration by parts formulas and, as a consequence, shows the shifted Harnack inequalities and the existence of the density with respect to the Lebesgue measure for solution.

2 Preliminaries

In this part, we recall some basic facts about fractional Brownian motion. For more details, one can refer to [22].

Let $B^H = \{B_t^H, t \in [0, T]\}$ be a d -dimensional fractional Brownian motion with Hurst parameter $H \in (0, 1)$ defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, i.e., B^H is a centered Gauss process with the covariance function $\mathbb{E}B_t^{H,i}B_s^{H,j} = R_H(t, s)\delta_{i,j}$, where

$$R_H(t, s) = \frac{1}{2} (t^{2H} + s^{2H} - |t - s|^{2H}).$$

In particular, if $H = \frac{1}{2}$, B^H is a d -dimensional Brownian motion. By the above covariance function, one can show that $\mathbb{E}|B_t^{H,i} - B_s^{H,i}|^p = C(p)|t - s|^{pH}$, $\forall p \geq 1$. As a consequence, $B_s^{H,i}$ have $(H - \epsilon)$ -order Hölder continuous paths for all $\epsilon > 0$, $i = 1, \dots, d$.

For each $t \in [0, T]$, we denote by \mathcal{F}_t the σ -algebra generated by the random variables $\{B_s^H : s \in [0, t]\}$ and the \mathbb{P} -null sets.

Let \mathcal{E} be the set of step functions on $[0, T]$ with values in \mathbb{R}^d and \mathcal{H} be the Hilbert space defined as the closure of \mathcal{E} with respect to the scalar product

$$\langle (I_{[0,t_1]}, \dots, I_{[0,t_d]}), (I_{[0,s_1]}, \dots, I_{[0,s_d]}) \rangle_{\mathcal{H}} = \sum_{i=1}^d R_H(t_i, s_i).$$

The mapping $(I_{[0,t_1]}, \dots, I_{[0,t_d]}) \mapsto \sum_{i=1}^d B_{t_i}^{H,i}$ can be extended to an isometry between \mathcal{H} and the Gauss space \mathcal{H}_1 associated with B^H . Denote this isometry by $\phi \mapsto B^H(\phi)$.

On the other hand, it follows by [10] that the covariance kernel $R_H(t, s)$ can be written as

$$R_H(t, s) = \int_0^{t \wedge s} K_H(t, r) K_H(s, r) dr,$$

where K_H is a square integrable kernel given by

$$K_H(t, s) = \Gamma(H + \frac{1}{2})^{-1} (t - s)^{H - \frac{1}{2}} F(H - \frac{1}{2}, \frac{1}{2} - H, H + \frac{1}{2}, 1 - \frac{t}{s}),$$

in which $F(\cdot, \cdot, \cdot, \cdot)$ is the Gauss hypergeometric function.

Define the linear operator $K_H^* : \mathcal{E} \rightarrow L^2([0, T], \mathbb{R}^d)$ as follows

$$(K_H^* \phi)(s) = K_H(T, s) \phi(s) + \int_s^T (\phi(r) - \phi(s)) \frac{\partial K_H}{\partial r}(r, s) dr.$$

In particular, $(K_H^*(I_{[0,t_1]}, \dots, I_{[0,t_d]}))(\cdot) = (K_H(t_1, \cdot), \dots, K_H(t_d, \cdot))$.

By [3], we know that, for all $\phi, \psi \in \mathcal{E}$, $\langle K_H^* \phi, K_H^* \psi \rangle_{L^2([0, T], \mathbb{R}^d)} = \langle \phi, \psi \rangle_{\mathcal{H}}$ holds. By the B.L.T. theorem, K_H^* can be extended to an isometry between \mathcal{H} and $L^2([0, T], \mathbb{R}^d)$. Hence, due to [3], there exists a Brownian motion W such that $B^H(\phi) = W(K_H^* \phi)$, $\forall \phi \in \mathcal{H}$.

According to [10], the operator $K_H : L^2([0, T], \mathbb{R}^d) \rightarrow I_{0+}^{H+\frac{1}{2}}(L^2([0, T], \mathbb{R}^d))$ associated with the kernel $K_H(\cdot, \cdot)$ is defined as follows

$$(K_H f^i)(t) := \int_0^t K_H(t, s) f^i(s) ds, \quad i = 1, \dots, d,$$

where $I_{0+}^{H+\frac{1}{2}}$ is the $(H + \frac{1}{2})$ -order left fractional Riemann-Liouville integral operator on $[0, T]$. It is an isomorphism and for each $f \in L^2([0, T], \mathbb{R}^d)$,

$$\begin{aligned} (K_H f)(s) &= I_{0+}^{2H} s^{\frac{1}{2}-H} I_{0+}^{\frac{1}{2}-H} s^{H-\frac{1}{2}} f, \quad H \leq \frac{1}{2}, \\ (K_H f)(s) &= I_{0+}^1 s^{H-\frac{1}{2}} I_{0+}^{H-\frac{1}{2}} s^{\frac{1}{2}-H} f, \quad H \geq \frac{1}{2}. \end{aligned}$$

Thus, for all $h \in I_{0+}^{H+\frac{1}{2}}(L^2([0, T], \mathbb{R}^d))$, the inverse operator K_H^{-1} is of the following form

$$\begin{aligned} (K_H^{-1} h)(s) &= s^{H-\frac{1}{2}} D_{0+}^{H-\frac{1}{2}} s^{\frac{1}{2}-H} h', \quad H > \frac{1}{2}, \\ (K_H^{-1} h)(s) &= s^{\frac{1}{2}-H} D_{0+}^{\frac{1}{2}-H} s^{H-\frac{1}{2}} D_{0+}^{2H} h, \quad H < \frac{1}{2}, \end{aligned}$$

where D_{0+}^α is α -order left-sided Riemann-Liouville derivative, $\alpha \in (0, 1)$. For more details on fractional calculus, one can refer to [27].

In particular, if h is absolutely continuous, we get

$$(K_H^{-1}h)(s) = s^{H-\frac{1}{2}} I_{0+}^{\frac{1}{2}-H} s^{\frac{1}{2}-H} h', \quad H < \frac{1}{2}.$$

In the paper, we are interested in the following stochastic differential equations driven by fractional Brownian motion on \mathbb{R}^d :

$$dX_t = b(X_t)dt + \sigma(X_t)dB_t^H, \quad X_0 = x, \quad (2.1)$$

where $b : \mathbb{R}^d \rightarrow \mathbb{R}^d$ and $\sigma : \mathbb{R}^d \rightarrow \mathbb{R}^d \times \mathbb{R}^d$.

Define $P_t f(x) := \mathbb{E}f(X_t^x)$, $t \in [0, T]$, $f \in \mathcal{B}_b(\mathbb{R}^d)$, where X_t^x is the solution to the equation (2.1) with the initial value x and $\mathcal{B}_b(\mathbb{R}^d)$ denotes the set of all bounded measurable functions on \mathbb{R}^d . For $0 < \lambda \leq 1$, we denote by $C^\lambda(0, T; \mathbb{R}^d)$ the space of λ -Hölder continuous functions $f : [0, T] \rightarrow \mathbb{R}^d$ equipped with the norm

$$|||f|||_\lambda := \|f\|_\infty + \|f\|_\lambda,$$

where $\|f\|_\infty = \sup_{0 \leq t \leq T} |f(t)|$ and $\|f\|_\lambda = \sup_{0 \leq s < t \leq T} \frac{|f(t) - f(s)|}{|t - s|^\lambda}$.

We will establish the Bismut derivative formula and the integration by parts formula for P_T , and moreover present some applications.

3 Bismut derivative formula

In this part, we fix $\frac{1}{2} < H < 1$ and deal with the equation (2.1) with additive noise. Assume the diffusion coefficient σ is equal to the unit matrix I_d .

Next we give some assumptions on the drift coefficient b :

(H1): b is differentiable, and $|\nabla b| \leq K_1$, $|\nabla b(x) - \nabla b(y)| \leq K_2|x - y|$, $\forall x, y \in \mathbb{R}^d$, where K_1 and K_2 are positive constants.

The aim of the part is to establish a Bismut derivative formula for P_T which will imply the Harnack inequality. For $f \in \mathcal{B}_b(\mathbb{R}^d)$, $x, y \in \mathbb{R}^d$, $T > 0$, we will consider

$$\nabla_y P_T f(x) := \lim_{\epsilon \rightarrow 0} \frac{P_T f(x + \epsilon y) - P_T f(x)}{\epsilon}.$$

Theorem 3.1 (Bismut derivative formula) Assume (H1). Then, for each $T > 0$, $f \in \mathcal{B}_b(\mathbb{R}^d)$, $x, y \in \mathbb{R}^d$, $\nabla_y P_T f(x)$ exists and satisfies

$$\nabla_y P_T f(x) = \mathbb{E}(f(X_T^x)N_T),$$

where

$$\begin{aligned} N_T = & \frac{1}{\Gamma(\frac{3}{2} - H)T} \left\{ \int_0^T \left\langle s^{\frac{1}{2}-H} [(T-s)\nabla_y b(X(s)) + y], dW_s \right\rangle \right. \\ & + (H - \frac{1}{2}) \int_0^T \left\langle s^{H-\frac{1}{2}} \int_0^s \frac{s^{\frac{1}{2}-H} - r^{\frac{1}{2}-H}}{(s-r)^{\frac{1}{2}+H}} [(T-r)\nabla_y b(X(r)) + y] dr, dW_s \right\rangle \\ & \left. - (H - \frac{1}{2}) \int_0^T \left\langle \int_0^s \frac{(T-r)\nabla_y b(X(r)) - (T-s)\nabla_y b(X(s))}{(s-r)^{\frac{1}{2}+H}} dr, dW_s \right\rangle \right\}. \end{aligned}$$

Proof. We restrict ourselves to the case $d = 1$ for simplicity.

The proof will be divided into two steps.

Step 1. For every $\epsilon > 0$ and $y \in \mathbb{R}$, let us introduce the following coupled stochastic differential equation

$$dX_t^\epsilon = b(X_t)dt + dB_t^H - \frac{\epsilon}{T}ydt, \quad X_0^\epsilon = x + \epsilon y, \quad (3.1)$$

$$dX_t = b(X_t)dt + dB_t^H, \quad X_0 = x. \quad (3.2)$$

Obviously the equation (3.1) has a unique solution. In view of (3.1) and (3.2), we conclude that $X_t^\epsilon - X_t = \frac{T-t}{T}\epsilon y$, $\forall t \in [0, T]$, in particular, $X_T^\epsilon = X_T$. Let $\eta_t = b(X_t) - b(X_t^\epsilon) - \frac{\epsilon}{T}y$ and $\tilde{B}_t^H = B_t^H + \int_0^t \eta_s ds$ $\forall t \in [0, T]$, then we can reformulate (3.1) as

$$dX_t^\epsilon = b(X_t^\epsilon)dt + d\tilde{B}_t^H.$$

Observe that, by the Gronwall lemma,

$$\|X\|_\infty \leq [(1 + K_1 T)|x| + T|b(x)| + \|B^H\|_\infty]e^{K_1 T},$$

and

$$\begin{aligned} |X_t - X_s| &\leq [K_1\|X\|_\infty + K_1|x| + |b(x)|]|t - s| + |B_t^H - B_s^H| \\ &\leq d_1|t - s| + d_2\|B^H\|_\infty|t - s| + |B_t^H - B_s^H|, \end{aligned} \quad (3.3)$$

where $d_1 = K_1 e^{K_1 T}|x| + (1 + K_1 T e^{K_1 T})(K_1|x| + |b(x)|)$ and $d_2 = K_1 e^{K_1 T}$.

Next we will show that \tilde{B}^H is a fractional Brownian motion under some probability space.

First note that $\int_0^\cdot \eta_r dr \in I_{0+}^{H+\frac{1}{2}}(L^2([0, T]))$.

In fact, by the definition of η , we have

$$\eta_t - \eta_s = b(X_t) - b(X_s) - (b(X_t^\epsilon) - b(X_s^\epsilon)).$$

Hence, it follows from the condition of $|b'(x)| \leq K_1$, $\forall x \in \mathbb{R}$ that

$$|\eta_t - \eta_s| \leq K_1(|X_t - X_s| + |X_t^\epsilon - X_s^\epsilon|) \leq K_1 \left(2|X_t - X_s| + \frac{|t - s|}{T}\epsilon|y| \right).$$

By [21, Theorem 2.1], it follows that X have α -order Hölder continuous paths for all $\alpha \in (0, H)$.

Therefore, the process η also have α -order Hölder continuous paths for all $\alpha \in (0, H)$. It implies

that $\eta \in I_{0+}^{H-\frac{1}{2}}(L^2([0, T]))$ and moreover $\int_0^\cdot \eta_r dr \in I_{0+}^{H+\frac{1}{2}}(L^2([0, T]))$.

According to the integral representation of fractional Brownian motion and the derived above fact, we deduce that

$$\tilde{B}_t^H = \int_0^t K_H(t, s) d\tilde{W}_s,$$

where $\tilde{W}_t = W_t + \int_0^t (K_H^{-1} \int_0^\cdot \eta_r dr)(s) ds$.

Let

$$R_\epsilon := \exp \left[- \int_0^T \left(K_H^{-1} \int_0^\cdot \eta_r dr \right) (s) dW_s - \frac{1}{2} \int_0^T \left(K_H^{-1} \int_0^\cdot \eta_r dr \right)^2 (s) ds \right].$$

Now we are in position to prove that $(\tilde{B}_t^H)_{0 \leq t \leq T}$ is an \mathcal{F}_t -fractional Brownian motion with Hurst parameter H under the new probability $R_\epsilon P$. Due to the Girsanov theorem for the fractional Brownian motion (see e.g. [10, Theorem 4.9] or [20, Theorem 2]), it suffices to show that $\mathbb{E}R_\epsilon = 1$.

Recalling the facts: in case of $H > \frac{1}{2}$, $\forall h \in I_{0+}^{H+\frac{1}{2}}(L^2([0, T]))$,

$$(K_H^{-1}h)(s) = s^{H-\frac{1}{2}}D_{0+}^{H-\frac{1}{2}}(s^{\frac{1}{2}-H}h'),$$

and $\forall f \in I_{0+}^\alpha(L^p)$, $0 < \alpha < 1$, $p > 1$,

$$D_{0+}^\alpha f(x) = \frac{1}{\Gamma(1-\alpha)} \left(\frac{f(x)}{x^\alpha} + \alpha \int_0^x \frac{f(x) - f(y)}{(x-y)^{1+\alpha}} dy \right),$$

we can get

$$\begin{aligned} \left(K_H^{-1} \int_0^\cdot \eta_r dr \right) (s) &= s^{H-\frac{1}{2}} D_{0+}^{H-\frac{1}{2}} (r^{\frac{1}{2}-H} \eta_r)(s) \\ &= \frac{1}{\Gamma(\frac{3}{2}-H)} \left(s^{\frac{1}{2}-H} \eta_s + (H - \frac{1}{2}) s^{H-\frac{1}{2}} \int_0^s \frac{s^{\frac{1}{2}-H} \eta_s - r^{\frac{1}{2}-H} \eta_r}{(s-r)^{\frac{1}{2}+H}} dr \right) \\ &= \frac{1}{\Gamma(\frac{3}{2}-H)} \left(s^{\frac{1}{2}-H} \eta_s + (H - \frac{1}{2}) s^{H-\frac{1}{2}} \int_0^s \frac{s^{\frac{1}{2}-H} - r^{\frac{1}{2}-H}}{(s-r)^{\frac{1}{2}+H}} \eta_r dr \right. \\ &\quad \left. + (H - \frac{1}{2}) \int_0^s \frac{\eta_s - \eta_r}{(s-r)^{\frac{1}{2}+H}} dr \right) \\ &=: \frac{1}{\Gamma(\frac{3}{2}-H)} (I_1 + I_2 + I_3). \end{aligned} \tag{3.4}$$

Notice that $|\eta_t| \leq (K_1 + \frac{1}{T})|y|\epsilon$, and

$$\int_0^s \frac{s^{\frac{1}{2}-H} - r^{\frac{1}{2}-H}}{(s-r)^{\frac{1}{2}+H}} dr = \int_0^1 \frac{\theta^{\frac{1}{2}-H} - 1}{(1-\theta)^{\frac{1}{2}+H}} d\theta \cdot s^{1-2H} =: C_0 s^{1-2H},$$

where we use the change of variables $\theta = \frac{r}{s}$ for the first integral and C_0 is a positive constant. What remains to be dealt with is the term I_3 . Let $F(x, y) := b(x) - b(y)$, we have

$$\begin{aligned} \int_0^s \frac{\eta_s - \eta_r}{(s-r)^{\frac{1}{2}+H}} dr &= \int_0^s \frac{F(X_r^\epsilon, X_s^\epsilon) - F(X_r, X_s)}{(s-r)^{\frac{1}{2}+H}} dr \\ &= \int_0^s \frac{\partial_1 F(\tilde{X}_r, \tilde{X}_s)(X_r^\epsilon - X_r) + \partial_2 F(\tilde{X}_r, \tilde{X}_s)(X_s^\epsilon - X_s)}{(s-r)^{\frac{1}{2}+H}} dr \\ &= \frac{\epsilon}{T} y \int_0^s \frac{b'(\tilde{X}_r)(T-r) - b'(\tilde{X}_s)(T-s)}{(s-r)^{\frac{1}{2}+H}} dr \\ &= \frac{\epsilon}{T} y \int_0^s \frac{(b'(\tilde{X}_r) - b'(\tilde{X}_s))(T-r) + b'(\tilde{X}_s)(s-r)}{(s-r)^{\frac{1}{2}+H}} dr, \end{aligned}$$

where we use the mean value theorem for the function F in the second relation and $\tilde{X}_r = X_r + \theta_1(X_r^\epsilon - X_r)$, $\tilde{X}_s = X_s + \theta_1(X_s^\epsilon - X_s)$, $0 < \theta_1 < 1$.

By (H1) and (3.3), we obtain

$$\left| \int_0^s \frac{\eta_s - \eta_r}{(s-r)^{\frac{1}{2}+H}} dr \right| \leq K_2 |y| \epsilon \int_0^s \frac{|\tilde{X}_s - \tilde{X}_r|}{(s-r)^{\frac{1}{2}+H}} dr + K_1 \frac{|y|}{T} \epsilon \int_0^s (s-r)^{\frac{1}{2}-H} dr$$

$$\begin{aligned}
&\leq K_2|y|\epsilon \int_0^s \frac{\frac{|y|}{T}\epsilon(s-r) + |X_s - X_r|}{(s-r)^{\frac{1}{2}+H}} dr + K_1 \frac{|y|}{T} \epsilon \int_0^s (s-r)^{\frac{1}{2}-H} dr \\
&\leq \left(\frac{Td_1K_2 + K_2|y|\epsilon + K_1}{\left(\frac{3}{2}-H\right)T} \right) |y|\epsilon s^{\frac{3}{2}-H} + \frac{d_2K_2}{\frac{3}{2}-H} |y|\epsilon \|B^H\|_\infty s^{\frac{3}{2}-H} \\
&\quad + \frac{K_2}{\frac{1}{2}-\delta} |y|\epsilon s^{\frac{1}{2}-\delta} \|B^H\|_{H-\delta},
\end{aligned}$$

where δ is taken such that $0 < \delta < \frac{1}{2}$.

Now, we can estimate $K_H^{-1}(\int_0^\cdot \eta_r dr)(s)$ as follows

$$\left| \left(K_H^{-1} \int_0^\cdot \eta_r dr \right) (s) \right| \leq C_1 \epsilon s^{\frac{1}{2}-H} + C_2 \epsilon s^{\frac{3}{2}-H} + C_3 \epsilon s^{\frac{3}{2}-H} \|B^H\|_\infty + C_4 \epsilon s^{\frac{1}{2}-\delta} \|B^H\|_{H-\delta},$$

where

$$\begin{aligned}
C_1 &= \frac{1}{\Gamma(\frac{3}{2}-H)} \left(1 + C_0 \left(H - \frac{1}{2} \right) \right) \left(K_1 + \frac{1}{T} \right) |y|, \\
C_2 &= \frac{1}{\Gamma(\frac{3}{2}-H)} \frac{H - \frac{1}{2}}{\frac{3}{2}-H} \frac{Td_1K_2 + K_2|y|\epsilon + K_1}{T} |y|, \\
C_3 &= \frac{1}{\Gamma(\frac{3}{2}-H)} \frac{H - \frac{1}{2}}{\frac{3}{2}-H} d_2K_2 |y|, \\
C_4 &= \frac{1}{\Gamma(\frac{3}{2}-H)} \frac{H - \frac{1}{2}}{\frac{1}{2}-\delta} K_2 |y|.
\end{aligned}$$

Moreover, the above estimate follows that

$$\begin{aligned}
&\int_0^T \left| \left(K_H^{-1} \int_0^\cdot \eta_r dr \right) (s) \right|^2 ds \\
&\leq 4\epsilon^2 \left(\frac{C_1^2 T^{2-2H}}{2-2H} + \frac{C_2^2 T^{4-2H}}{4-2H} + \frac{C_3^2 T^{4-2H}}{4-2H} \|B^H\|_\infty^2 + \frac{C_4^2 T^{2-2\delta}}{2-2\delta} \|B^H\|_{H-\delta}^2 \right). \quad (3.5)
\end{aligned}$$

As a consequence, we obtain

$$\begin{aligned}
\mathbb{E} \exp \left[\frac{1}{2} \int_0^T \left(K_H^{-1} \int_0^\cdot \eta_r dr \right)^2 (s) ds \right] &\leq C_5 \mathbb{E} \exp \left[\epsilon^2 (C_6 \|B^H\|_\infty^2 + C_7 \|B^H\|_{H-\delta}^2) \right] \\
&\leq C_5 \mathbb{E} \exp \left[(C_6 \vee C_7) \epsilon^2 \|B^H\|_{H-\delta}^2 \right] \quad (3.6)
\end{aligned}$$

where $C_5 = \exp \left[2\epsilon^2 \left(\frac{C_1^2 T^{2-2H}}{2-2H} + \frac{C_2^2 T^{4-2H}}{4-2H} \right) \right]$, $C_6 = \frac{C_3^2 T^{4-2H}}{2-H}$ and $C_7 = \frac{C_4^2 T^{2-2\delta}}{1-\delta}$.

Using the Fernique theorem, we know that, the right-hand side of (3.6) is finite when ϵ is small enough. Consequently, the Novikov condition yields that $\mathbb{E}R_\epsilon = 1$.

Step 2. From step 1, we have already known that, for small enough ϵ , $(\tilde{B}_t^H)_{0 \leq t \leq T}$ is an \mathcal{F}_t -fractional Brownian motion with Hurst parameter H under the probability $R_\epsilon P$. So, under $R_\epsilon P$, the law of the process $(X_t^{\epsilon, x+\epsilon y})_{t \in [0, T]}$ is the same as $(X_t^{x+\epsilon y})_{t \in [0, T]}$ under P , where the superscript $x + \epsilon y$ stands for starting part of the corresponding equation. By the fact: $X_T^{\epsilon, x+\epsilon y} = X_T^x$, we get

$$P_T f(x + \epsilon y) = \mathbb{E} f(X_T^{x+\epsilon y}) = \mathbb{E}(R_\epsilon f(X_T^{\epsilon, x+\epsilon y})) = \mathbb{E}(R_\epsilon f(X_T^x)). \quad (3.7)$$

Hence, (3.7) implies that

$$\nabla_y P_T f(x) := \lim_{\epsilon \rightarrow 0} \frac{P_T f(x + \epsilon y) - P_T f(x)}{\epsilon} = \lim_{\epsilon \rightarrow 0} \left[\mathbb{E} f(X_T^x) \frac{R_\epsilon - 1}{\epsilon} \right]. \quad (3.8)$$

Define $M_t := -\int_0^t (K_H^{-1} \int_0^\cdot \eta_r dr) (s) dW_s$, then $R_\epsilon = \exp[M_T - \frac{1}{2}\langle M \rangle_T]$. Next we will show that

$$\lim_{\epsilon \rightarrow 0} \mathbb{E} \left(f(X_T^x) \frac{R_\epsilon - 1}{\epsilon} \right) = \lim_{\epsilon \rightarrow 0} \mathbb{E} \left(f(X_T^x) \frac{M_T - \frac{1}{2}\langle M \rangle_T}{\epsilon} \right). \quad (3.9)$$

Since f is bounded, it suffices to prove that

$$\lim_{\epsilon \rightarrow 0} \mathbb{E} \frac{R_\epsilon - 1}{\epsilon} = \lim_{\epsilon \rightarrow 0} \mathbb{E} \frac{M_T - \frac{1}{2}\langle M \rangle_T}{\epsilon}.$$

Without lost of generality, in the remaining proof, we suppose that $\epsilon \leq 1$.

Indeed, noting that, for all $x \in \mathbb{R}$, $|e^x - 1 - x| \leq x^2 e^{|x|}$ and $x^2 \leq e^{|x|}$ hold, we deduce that

$$\begin{aligned} \left| \frac{R_\epsilon - 1 - (M_T - \frac{1}{2}\langle M \rangle_T)}{\epsilon} \right| &\leq \frac{1}{\epsilon} \left(M_T - \frac{1}{2}\langle M \rangle_T \right)^2 e^{|M_T| + \frac{1}{2}\langle M \rangle_T} \\ &= \epsilon^{\frac{1}{2}} \left(\frac{1}{\epsilon^{3/4}} M_T - \frac{1}{2\epsilon^{3/4}} \langle M \rangle_T \right)^2 e^{|M_T| + \frac{1}{2}\langle M \rangle_T} \\ &\leq \epsilon^{\frac{1}{2}} e^{\frac{1}{4}\epsilon^{3/4}|M_T| + \frac{1}{8}\epsilon^{3/4}\langle M \rangle_T} e^{|M_T| + \frac{1}{2}\langle M \rangle_T} \\ &\leq \epsilon^{\frac{1}{2}} e^{\frac{2}{\epsilon^{3/4}}|M_T| + \frac{1}{\epsilon^{3/4}}\langle M \rangle_T} \\ &\leq \epsilon^{\frac{1}{2}} \left(e^{\frac{2}{\epsilon^{3/4}}M_T} + e^{\frac{-2}{\epsilon^{3/4}}M_T} \right) e^{\frac{1}{\epsilon^{3/4}}\langle M \rangle_T} \\ &= \epsilon^{\frac{1}{2}} \left(e^{\frac{2}{\epsilon^{3/4}}M_T - \frac{4}{\epsilon^{3/2}}\langle M \rangle_T} + e^{\frac{-2}{\epsilon^{3/4}}M_T - \frac{4}{\epsilon^{3/2}}\langle M \rangle_T} \right) e^{\frac{1}{\epsilon^{3/4}}\langle M \rangle_T + \frac{4}{\epsilon^{3/2}}\langle M \rangle_T}. \end{aligned}$$

Using the Hölder inequality, the C_r inequality and (3.5), the above inequality yields that

$$\begin{aligned} &\mathbb{E} \left| \frac{R_\epsilon - 1 - (M_T - \frac{1}{2}\langle M \rangle_T)}{\epsilon} \right| \\ &\leq \epsilon^{\frac{1}{2}} \left(2\mathbb{E} e^{\frac{4}{\epsilon^{3/4}}M_T - \frac{8}{\epsilon^{3/2}}\langle M \rangle_T} + 2\mathbb{E} e^{\frac{-4}{\epsilon^{3/4}}M_T - \frac{8}{\epsilon^{3/2}}\langle M \rangle_T} \right)^{\frac{1}{2}} \left(\mathbb{E} e^{\frac{2}{\epsilon^{3/4}}\langle M \rangle_T + \frac{8}{\epsilon^{3/2}}\langle M \rangle_T} \right)^{\frac{1}{2}} \\ &= (4\epsilon)^{\frac{1}{2}} \left(\mathbb{E} e^{\frac{2}{\epsilon^{3/4}}\langle M \rangle_T + \frac{8}{\epsilon^{3/2}}\langle M \rangle_T} \right)^{\frac{1}{2}}, \end{aligned}$$

which shows that (3.9) is true.

Noting that (3.5) and $\mathbb{E}(\|B^H\|_\infty^2 + \|B^H\|_{H-\delta}^2) < \infty$, we may write (3.9) as follows

$$\lim_{\epsilon \rightarrow 0} \mathbb{E} \left(f(X_T^x) \frac{R_\epsilon - 1}{\epsilon} \right) = \lim_{\epsilon \rightarrow 0} \mathbb{E} \left(f(X_T^x) \frac{M_T}{\epsilon} \right). \quad (3.10)$$

In view of (3.4), M_T can be decomposable as

$$\begin{aligned} M_T &= -\int_0^T \left(K_H^{-1} \int_0^\cdot \eta_r dr \right) (s) dW_s \\ &= \frac{1}{\Gamma(\frac{3}{2} - H)} \left[\int_0^T s^{\frac{1}{2}-H} (-\eta_s) dW_s + (-1)(H - \frac{1}{2}) \int_0^T s^{H-\frac{1}{2}} \int_0^s \frac{s^{\frac{1}{2}-H} - r^{\frac{1}{2}-H}}{(s-r)^{\frac{1}{2}+H}} \eta_r dr dW_s \right] \end{aligned}$$

$$\begin{aligned}
& +(-1)(H - \frac{1}{2}) \int_0^T \int_0^s \frac{\eta_s - \eta_r}{(s-r)^{\frac{1}{2}+H}} dr dW_s \Big] \\
& =: \frac{1}{\Gamma(\frac{3}{2}-H)} [J_1 + J_2 + J_3].
\end{aligned} \tag{3.11}$$

For the term J_1 , making use of the Jensen inequality, we obtain

$$\begin{aligned}
& \mathbb{E} \left| \int_0^T \frac{s^{\frac{1}{2}-H}(-\eta_s)}{\epsilon} dW_s - \int_0^T s^{\frac{1}{2}-H} (1 + b'(X_s)(T-s)) \frac{y}{T} dW_s \right| \\
& \leq \left[\mathbb{E} \int_0^T s^{1-2H} \left(\frac{b(X_s^\epsilon) - b(X_s) - b'(X_s) \frac{T-s}{T} \epsilon y}{\epsilon} \right)^2 ds \right]^{\frac{1}{2}} \\
& = \left[\mathbb{E} \int_0^T s^{1-2H} (b'(X_s + \theta_2(X_s^\epsilon - X_s)) - b'(X_s))^2 \left(\frac{T-s}{T} y \right)^2 ds \right]^{\frac{1}{2}},
\end{aligned} \tag{3.12}$$

where we use the mean value theorem in the last expression and $\theta_2 \in (0, 1)$.

The dominated convergence theorem implies that the right-hand side of (3.12) goes to 0 as ϵ tends to 0.

Similar to the treatment of J_1 , for J_2 , we conclude that

$$\mathbb{E} \left| \int_0^T s^{H-\frac{1}{2}} \int_0^s \frac{s^{\frac{1}{2}-H} - r^{\frac{1}{2}-H}}{(s-r)^{\frac{1}{2}+H}} \left[\frac{-\eta_r}{\epsilon} - (1 + b'(X_r)(T-r)) \frac{y}{T} \right] dr dW_s \right| \rightarrow 0, \tag{3.13}$$

as ϵ tends to 0.

For the term J_3 , we get

$$\begin{aligned}
& \mathbb{E} \left| \int_0^T \int_0^s \frac{\eta_s - \eta_r}{(s-r)^{\frac{1}{2}+H}} \frac{1}{\epsilon} dr dW_s - \int_0^T \int_0^s \frac{b'(X_r)(T-r) - b'(X_s)(T-s)}{(s-r)^{\frac{1}{2}+H}} \frac{y}{T} dr dW_s \right| \\
& \leq \left\{ \mathbb{E} \int_0^T \left| \int_0^s \frac{b(X_s) - b(X_r) - [b(X_s^\epsilon) - b(X_r^\epsilon)] - [b'(X_r)(T-r) - b'(X_s)(T-s)] \frac{\epsilon y}{T}}{\epsilon(s-r)^{\frac{1}{2}+H}} dr \right|^2 ds \right\}^{\frac{1}{2}} \\
& = \left\{ \mathbb{E} \int_0^T \left| \int_0^s \frac{[b'(\bar{X}_r) - b'(X_r) - b'(\bar{X}_s) + b'(X_s)](T-r) + [b'(\bar{X}_s) - b'(X_s)](s-r)}{(s-r)^{\frac{1}{2}+H}} \frac{y}{T} dr \right|^2 ds \right\}^{\frac{1}{2}},
\end{aligned} \tag{3.14}$$

where we apply the mean value theorem in the last relation and $\bar{X}_r = X_r + \theta_3(X_r^\epsilon - X_r)$, $\bar{X}_s = X_s + \theta_3(X_s^\epsilon - X_s)$, $\theta_3 \in (0, 1)$.

We first claim that

$$\lim_{\epsilon \rightarrow 0} \int_0^s \frac{[b'(\bar{X}_r) - b'(X_r) - b'(\bar{X}_s) + b'(X_s)](T-r) + [b'(\bar{X}_s) - b'(X_s)](s-r)}{(s-r)^{\frac{1}{2}+H}} dr = 0. \tag{3.15}$$

In fact, noting that $\bar{X}_s - \bar{X}_r = X_s - X_r + \theta_3 \frac{r-s}{T} \epsilon y$, by (3.3) we get

$$\frac{|b'(\bar{X}_r) - b'(X_r) - b'(\bar{X}_s) + b'(X_s)|}{(s-r)^{\frac{1}{2}+H}}$$

$$\begin{aligned}
&\leq K_2 \frac{|\bar{X}_s - \bar{X}_r| + |X_s - X_r|}{(s-r)^{\frac{1}{2}+H}} \\
&\leq K_2 \frac{2|X_s - X_r| + \frac{s-r}{T}|y|}{(s-r)^{\frac{1}{2}+H}} \\
&\leq K_2 \frac{2d_1(s-r) + 2d_2\|B^H\|_\infty(s-r) + 2|B_s^H - B_r^H| + \frac{s-r}{T}|y|}{(s-r)^{\frac{1}{2}+H}} \\
&\leq K_2 \left(2d_1 + \frac{|y|}{T} + 2d_2\|B^H\|_\infty \right) (s-r)^{\frac{1}{2}-H} + 2K_2C(\omega)(s-r)^{\beta-\frac{1}{2}-H},
\end{aligned}$$

where we use the fact: B^H have β -order Hölder continuous trajectories for all $0 < \beta < H$ and choose $\frac{1}{2} < \beta < H$ here.

Consequently, (3.15) follows from the dominated convergence theorem.

In order to prove (3.14) converges to 0, as ϵ tends to 0, combined with (3.15), it suffices to give a majorizing function due to the dominated convergence theorem.

Notice that

$$\begin{aligned}
&\left| \int_0^s \frac{[b'(\bar{X}_r) - b'(X_r) - b'(\bar{X}_s) + b'(X_s)](T-r) + [b'(\bar{X}_s) - b'(X_s)](s-r)}{(s-r)^{\frac{1}{2}+H}} \frac{y}{T} dr \right|^2 \\
&\leq 2y^2 \left| \int_0^s \left[K_2 \left(2d_1 + \frac{|y|}{T} + 2d_2\|B^H\|_\infty \right) (s-r)^{\frac{1}{2}-H} + 2K_2 \frac{|B_s^H - B_r^H|}{(s-r)^{\frac{1}{2}+H}} \right] dr \right|^2 \\
&\quad + 2 \left(2K_1 \frac{y}{T} \right)^2 \left| \int_0^s (s-r)^{\frac{1}{2}-H} dr \right|^2 \\
&\leq C_8 s^{3-2H} + C_9 s^{3-2H} \|B^H\|_\infty^2 + C_{10} s^{1-2\delta} \|B^H\|_{H-\delta}^2,
\end{aligned}$$

where C_8 , C_9 and C_{10} are positive constants.

Due to (3.12), (3.13) and (3.14), the proof is complete.

Remark 3.2 The Bismut derivative formula presented in Theorem 3.1 can be easily extended to the following equation

$$dX_t = b(X_t)dt + \sigma(t)dB_t^H, \quad X_0 = x, \quad t \in [0, T],$$

where the stochastic integral exists pathwise under proper assumptions.

As applications of the Bismut derivative formula derived above, we may get the explicit gradient estimate and the dimensional free Harnack inequality for P_T .

By the Fernique theorem, there exists a positive constant λ_0 such that $B_0 := \mathbb{E} \exp[\lambda_0 \|B^H\|_{H-\delta}^2] < \infty$. We set $\|\nabla P_T f(x)\| := \sup_{y \in \mathbb{R}^d, |y| \leq 1} |\nabla_y P_T f(x)|$, $\forall x \in \mathbb{R}^d$.

Corollary 3.3 Under assumptions of Theorem 3.1. Let $f \in \mathcal{B}_b(\mathbb{R}^d)$ be positive. Put

$$\begin{aligned}
A_1 &:= \frac{3}{(\Gamma(\frac{3}{2}-H))^2 T^{2H}} \left\{ \frac{(1+K_1 T)^2}{2-2H} + \frac{[C_0(H-\frac{1}{2})(1+K_1 T)]^2}{2-2H} + \left[\frac{\sqrt{3}(H-\frac{1}{2})(d_1 K_2 T + K_1)T}{\sqrt{4-2H}(\frac{3}{2}-H)} \right]^2 \right\}, \\
A_2 &:= \frac{9(H-\frac{1}{2})^2}{(\Gamma(\frac{3}{2}-H))^2} \left\{ \left[\left(\frac{d_2 K_2}{\frac{3}{2}-H} \right)^2 \frac{T^{4-2H}}{4-2H} \right] \vee \left[\left(\frac{K_2}{\frac{1}{2}-\delta} \right)^2 \frac{T^{2-2\delta}}{2-2\delta} \right] \right\}.
\end{aligned}$$

(1) For each $x \in \mathbb{R}^d$ and $\alpha \geq \sqrt{\frac{2A_2}{\lambda_0}}$, we have

$$\|\nabla P_T f(x)\| \leq \alpha [P_T(f \log f)(x) - (P_T f)(x)(\log P_T f)(x)] + \frac{1}{\alpha} \left(A_1 + \frac{A_2}{\lambda_0} \log B_0 \right) P_T f(x).$$

(2) Let $p > 1$. Then, for all $x, y \in \mathbb{R}^d$ such that $|x - y| \leq \frac{p-1}{p} \sqrt{\frac{\lambda_0}{2A_2}}$, we get

$$(P_T f(x))^p \leq P_T f^p(y) \exp \left[\frac{p}{p-1} \left(A_1 + \frac{A_2}{\lambda_0} \log B_0 \right) |x - y|^2 \right].$$

(3) For each $x \in \mathbb{R}^d$, $\lim_{|y-x| \rightarrow 0} P_T f(y) = P_T f(x)$, i.e. P_T is strong Feller.

Proof Let $f \in \mathcal{B}_b(\mathbb{R}^d)$ be positive. Combining Theorem 3.1 with the Young inequality (see e.g. [5, Lemma 2.4]) yield that, for each $\alpha > 0$,

$$|\nabla_y P_T f(x)| \leq \alpha [P_T(f \log f)(x) - (P_T f)(x)(\log P_T f)(x)] + [\alpha \log \mathbb{E} e^{\frac{1}{\alpha} N_T}] P_T f(x). \quad (3.16)$$

Now we turn to calculate $\mathbb{E} e^{\frac{1}{\alpha} N_T}$.

By the expression of N_T , we obtain

$$\begin{aligned} \langle N \rangle_T &\leq \frac{3}{[\Gamma(\frac{3}{2} - H)T]^2} \times \\ &\quad \left\{ \int_0^T s^{1-2H} |(T-s)\nabla_y b(X(s)) + y|^2 ds \right. \\ &\quad + (H - \frac{1}{2})^2 \int_0^T s^{2H-1} \left| \int_0^s \frac{s^{\frac{1}{2}-H} - r^{\frac{1}{2}-H}}{(s-r)^{\frac{1}{2}+H}} [(T-r)\nabla_y b(X(r)) + y] dr \right|^2 ds \\ &\quad \left. + (H - \frac{1}{2})^2 \int_0^T \left| \int_0^s \frac{(T-r)\nabla_y b(X(r)) - (T-s)\nabla_y b(X(s))}{(s-r)^{\frac{1}{2}+H}} dr \right|^2 ds \right\} \\ &=: \frac{3}{[\Gamma(\frac{3}{2} - H)T]^2} \{I_4 + I_5 + I_6\}. \end{aligned} \quad (3.17)$$

A simple calculus shows that

$$\begin{aligned} I_4 &\leq \frac{[(1 + K_1 T)]^2}{2 - 2H} T^{2-2H} |y|^2, \\ I_5 &\leq \frac{[C_0(H - \frac{1}{2})(1 + K_1 T)]^2}{2 - 2H} T^{2-2H} |y|^2, \\ I_6 &\leq \frac{3}{4 - 2H} \left[\frac{(H - \frac{1}{2})(d_1 K_2 T + K_1)}{\frac{3}{2} - H} \right]^2 T^{4-2H} |y|^2 + \frac{3}{4 - 2H} \left[\frac{(H - \frac{1}{2})d_2 K_2}{\frac{3}{2} - H} \right]^2 T^{6-2H} \|B^H\|_\infty^2 |y|^2 \\ &\quad + \frac{3}{2 - 2\delta} \left[\frac{(H - \frac{1}{2})K_2}{\frac{1}{2} - \delta} \right]^2 T^{4-2\delta} \|B^H\|_{H-\delta}^2 |y|^2. \end{aligned}$$

Therefore, we get

$$\langle N \rangle_T \leq (A_1 + A_2 \|B^H\|_{H-\delta}^2) |y|^2,$$

Then, for any $\alpha \geq \sqrt{\frac{2A_2}{\lambda_0}}|y| =: \alpha_0$, we have

$$\begin{aligned}
\mathbb{E} \exp \left[\frac{1}{\alpha} N_T \right] &\leq \left(\mathbb{E} \exp \left[\frac{2}{\alpha^2} \langle N \rangle_T \right] \right)^{\frac{1}{2}} \\
&\leq \exp \left[\frac{A_1}{\alpha^2} |y|^2 \right] \left(\mathbb{E} \exp \left[\frac{2A_2}{\alpha_0^2} |y|^2 \|B^H\|_{H-\delta}^2 \right] \right)^{\frac{\alpha_0^2}{2\alpha^2}} \\
&= \exp \left[\frac{A_1}{\alpha^2} |y|^2 \right] \left(\mathbb{E} \exp [\lambda_0 \|B^H\|_{H-\delta}^2] \right)^{\frac{\alpha_0^2}{2\alpha^2}} \\
&= \exp \left[\left(A_1 + \frac{A_2}{\lambda_0} \log B_0 \right) \frac{|y|^2}{\alpha^2} \right]. \tag{3.18}
\end{aligned}$$

(1) By (3.16) and (3.18), we get, for $\alpha \geq \sqrt{\frac{2A_2}{\lambda_0}}|y|$,

$$|\nabla_y P_T f(x)| \leq \alpha [P_T(f \log f)(x) - (P_T f)(x)(\log P_T f)(x)] + \frac{1}{\alpha} \left(A_1 + \frac{A_2}{\lambda_0} \log B_0 \right) |y|^2 P_T f(x).$$

Let $|y| \leq 1$, we derive the desired result.

(2) The part of the proof follows the arguments of [5, Theorem 1.2]. Let $\beta(s) = 1+s(p-1)$, $\gamma(s) = x + s(y-x)$, $s \in [0, 1]$. We obtain

$$\begin{aligned}
&\frac{d}{ds} \log(P_T f^{\beta(s)})^{\frac{p}{\beta(s)}}(\gamma(s)) \\
&= \frac{p}{\beta(s) P_T f^{\beta(s)}(\gamma(s))} \times \\
&\left\{ \frac{p-1}{\beta(s)} \left[P_T(f^{\beta(s)} \log f^{\beta(s)}) - (P_T f^{\beta(s)}) \log P_T f^{\beta(s)} \right] (\gamma(s)) - \nabla_{y-x} P_T f^{\beta(s)}(\gamma(s)) \right\}.
\end{aligned}$$

Then, for $x, y \in \mathbb{R}^d$ satisfying $|x-y| \leq \frac{p-1}{p} \sqrt{\frac{\lambda_0}{2A_2}}$, it follows from (3.16) and (3.18) that

$$\begin{aligned}
&\frac{d}{ds} \log(P_T f^{\beta(s)})^{\frac{p}{\beta(s)}}(\gamma(s)) \\
&\geq -\frac{p}{\beta(s)\alpha} \left(A_1 + \frac{A_2}{\lambda_0} \log B_0 \right) |y-x|^2 \\
&= -\frac{p}{p-1} \left(A_1 + \frac{A_2}{\lambda_0} \log B_0 \right) |y-x|^2,
\end{aligned}$$

where we take $\alpha = \frac{p-1}{\beta(s)}$ in (3.16).

Integrating on the interval $[0, 1]$ with respect to ds , we get the desired result.

(3) According to the proof of [9, Proposition 4.1], the above result (2) implies the strong Feller property of P_T .

Remark 3.4 *The Harnack inequality in Corollary 3.3 is local in the sense that $|x-y|$ is bounded by a constant. How to establish a global Harnack inequality is an interesting problem.*

The Harnack inequality for one-dimensional equations with multiplicative noises can be derived from Corollary 3.3. Now we assume that $d = 1$ and consider the stochastic differential equation

$$dX_t = b(X_t)dt + \sigma(X_t)dB_t^H, \quad X_0 = x, \quad t \in [0, T]. \tag{3.19}$$

We give the following assumptions on the coefficients: (H2)

- (i) b and σ are both twice differentiable, and the functions themselves and their derivatives are bounded;
- (ii) There exist two constants d_3 and d_4 such that $d_4 > d_3 > 0$ and $d_3 \leq \sigma(x) \leq d_4$, $\forall x \in \mathbb{R}$.

Under the above assumptions, [21] shows that there exists a unique adapted solution to equation (3.19) whose paths are Hölder continuous of order $H - \epsilon$ for every $\epsilon > 0$. Note that the stochastic integral appears in equation (3.19) can be considered as pathwise integral. See [35] for more details. Set $\bar{P}_T f(x) := \mathbb{E}f(X_T^x)$, where $(X_t^x)_{t \in [0, T]}$ is the solution to equation (3.19) with initial value x .

Theorem 3.5 *Assume (H2) holds. Let $p > 1$ and $f \in \mathcal{B}_b(\mathbb{R})$ be positive function. Then, there exists two positive constants \bar{A}_1 and \bar{A}_2 such that the Harnack inequality*

$$(\bar{P}_T f(x))^p \leq \bar{P}_T f^p(y) \exp \left[\frac{p}{p-1} \frac{1}{d_3^2} \left(\bar{A}_1 + \frac{\bar{A}_2}{\lambda_0} \log B_0 \right) |x - y|^2 \right]$$

holds for all $x, y \in \mathbb{R}$ satisfying $|x - y| \leq \frac{p-1}{p} d_3 \sqrt{\frac{\lambda_0}{2A_2}}$.

Proof Put $F(y) := \int_0^y \frac{1}{\sigma(z)} dz$. Then, according to the change-of-variables formula [35, Theorem 4.3.1], we know that X is the unique solution to equation (3.19) if and only if the process $Y = F(X)$ is the unique solution of

$$dY_t = \frac{b(F^{-1}(Y_t))}{\sigma(F^{-1}(Y_t))} dt + dB_t^H, \quad Y_0 = F(x), \quad t \in [0, T]. \quad (3.20)$$

Define $\tilde{P}_T g(z) := \mathbb{E}g(Y_T^z)$, where $(Y_t^z)_{t \in [0, T]}$ is the solution of equation (3.20) with initial value z .

Note that $\frac{d}{dx} F^{-1}(x) = \sigma(F^{-1}(x))$. Hence, due to the assumption, we deduce that the coefficient $\frac{b(F^{-1}(x))}{\sigma(F^{-1}(x))}$ is twice differentiable and moreover,

$$\begin{aligned} \left(\frac{b(F^{-1}(x))}{\sigma(F^{-1}(x))} \right)' &= \frac{b'\sigma - b\sigma'}{\sigma^2}(F^{-1}(x)), \\ \left(\frac{b(F^{-1}(x))}{\sigma(F^{-1}(x))} \right)'' &= \frac{b''\sigma^2 - b\sigma\sigma'' - b'\sigma\sigma' + b\sigma'^2}{\sigma^3}(F^{-1}(x)) \end{aligned}$$

are both bounded. Then, by (2) of Corollary 3.3, we conclude that there exist two positive constants \bar{A}_1 and \bar{A}_2 such that, for each $g \in \mathcal{B}_b(\mathbb{R})$,

$$(\tilde{P}_T g(z_1))^p \leq \tilde{P}_T g^p(z_2) \exp \left[\frac{p}{p-1} \left(\bar{A}_1 + \frac{\bar{A}_2}{\lambda_0} \log B_0 \right) |z_1 - z_2|^2 \right] \quad (3.21)$$

holds for all $z_1, z_2 \in \mathbb{R}$ satisfying $|z_1 - z_2| \leq \frac{p-1}{p} \sqrt{\frac{\lambda_0}{2A_2}}$.

For $x, y \in \mathbb{R}$ such that $|x - y| \leq \frac{p-1}{p} d_3 \sqrt{\frac{\lambda_0}{2A_2}}$ and $f \in \mathcal{B}_b(\mathbb{R})$, we take $z_1 = F(x), z_2 = F(y)$ and $g = f \circ F^{-1}$. Then applying (3.21) to the particular choices z_1, z_2 and g , we get the desired result. Indeed, first we have

$$|z_1 - z_2| = |F(x) - F(y)| \leq \frac{1}{d_3} |x - y| \leq \frac{p-1}{p} \sqrt{\frac{\lambda_0}{2A_2}}.$$

Furthermore note that $Y_t^{F(x)} = F(X_t^x)$, $t \in [0, T]$. Then, we obtain

$$\tilde{P}_T g(z_1) = \mathbb{E}g(Y_T^{z_1}) = \mathbb{E}f \circ F^{-1}(F(X_T^x)) = \mathbb{E}f(X_T^x) = \bar{P}_T f(x)$$

and

$$\tilde{P}_T g^p(z_2) = \mathbb{E}g^p(F(X_T^y)) = \mathbb{E}[f \circ F^{-1}(F(X_T^y))]^p = \mathbb{E}f^p(X_T^y) = \bar{P}_T f^p(y),$$

which show that the proof is complete.

Note that the solution X of equation (2.1) is not a Markov process. Consequently, $(P_T)_{T \geq 0}$ does not consist of a semigroup. So, we introduce the following semigroup in discrete time, i.e. for any Borel set A in \mathbb{R}^d ,

$$P_T(x, A) := P_T I_A(x), \quad P_T^n(x, A) := \int_{\mathbb{R}^d} P_T^{n-1}(x, dy) P_T(y, A), \quad n \geq 2.$$

In general, $(P_T^n f)(x) = \int_{\mathbb{R}^d} f(y) P_T^n(x, dy)$, $x \in \mathbb{R}^d$, $f \in \mathcal{B}_b(\mathbb{R}^d)$.

Next let $\sigma = I_d$ in the equation (2.1) and we will concern with the existence of invariant probability measure for semigroup $(P_T^n)_{n \geq 1}$ for fixed $T > 0$. Besides the assumption (H1), we will introduce another important condition.

(H3): $\langle x, b(x) \rangle \leq K_3 |x|^2$, where $K_3 \in \mathbb{R}$ satisfies that $C_{12} e^{2K_3 T} < 1$, C_{12} is a positive constant given in the proof below.

Theorem 3.6 Assume (H1) and (H3). Then, the semigroup $(P_T^n)_{n \geq 1}$ has an invariant probability measure.

Proof We will make use of Krylov-Bogoliubov's method.

Let $x_0 \in \mathbb{R}^d$ and define

$$\mu_n := \frac{\sum_{k=1}^n \delta_{x_0} P_T^k}{n}, \quad n \geq 1,$$

i.e. for each $f \in \mathcal{B}_b(\mathbb{R}^d)$, $\mu_n(f) = \frac{\sum_{k=1}^n P_T^k f(x_0)}{n}$.

Next we will prove the tightness of $\{\mu_n\}_{n \geq 1}$.

By the change-of-variables formula [35, Theorem 4.3.1] and (H3), we get

$$\begin{aligned} |X_T^x|^2 &= |x|^2 + 2 \int_0^T \langle X_t^x, b(X_t^x) \rangle dt + 2 \int_0^T \langle X_t^x, dB_t^H \rangle \\ &\leq |x|^2 + 2K_3 \int_0^T |X_t^x|^2 dt + 2 \int_0^T \langle X_t^x, dB_t^H \rangle. \end{aligned} \quad (3.22)$$

According to [28, Lemma 5], we get the estimation of pathwise integral in (3.22) as follows

$$\left| \int_0^T \langle X_t^x, dB_t^H \rangle \right| \leq \frac{L}{\beta - \frac{1}{2}} \|B^H\|_\beta \left(\|X\|_\infty T^\beta + \|X\|_\beta T^{2\beta} \right), \quad (3.23)$$

where $L(> 0)$ and $\beta \in (\frac{1}{2}, H)$ are two constants.

By (3.3), we easily get

$$\|X\|_\infty \leq |x|^2 + \left(\frac{(1 + K_1 T)^2 e^{2K_1 T}}{4} + |b(x)| T e^{K_1 T} \right) + e^{K_1 T} \|B^H\|_\infty,$$

$$\|X\|_\beta \leq d_1 T^{1-\beta} + d_2 T^{1-\beta} \|B^H\|_\infty + \|B^H\|_\beta. \quad (3.24)$$

Substituting (3.23) and (3.24) into (3.22) and taking expectation give

$$\mathbb{E}|X_T^x|^2 \leq C_{11} + C_{12}|x|^2 + 2K_3 \int_0^T \mathbb{E}|X_t^x|^2 dt,$$

where

$$\begin{aligned} C_{11} &= \frac{2LT^\beta}{\beta - \frac{1}{2}} \left[\left(d_1 T + \frac{(1 + K_1 T)^2 e^{2K_1 T}}{4} + |b(x)| T e^{K_1 T} \right) \mathbb{E}\|B^H\|_\beta \right. \\ &\quad \left. + (e^{K_1 T} + d_2 T) \mathbb{E}(\|B^H\|_\infty \|B^H\|_\beta) + T^\beta \mathbb{E}\|B^H\|_\beta^2 \right], \\ C_{12} &= 1 + \frac{2LT^\beta}{\beta - \frac{1}{2}} \mathbb{E}\|B^H\|_\beta. \end{aligned}$$

The Gronwall lemma yields

$$\mathbb{E}|X_T^x|^2 \leq (C_{11} + C_{12}|x|^2) e^{2K_3 T} =: C_{13} + C_{14}|x|^2. \quad (3.25)$$

Now we will show that $\{\int_{\mathbb{R}^d} |x|^2 P_T^n(x_0, dx)\}_{n \geq 1}$ is bounded by using an induction argument.

When $n = 1$, it follows from (3.25) that $\int_{\mathbb{R}^d} |x|^2 P_T(x_0, dx) \leq C_{13} + C_{14}|x_0|^2$. Suppose that

$$\int_{\mathbb{R}^d} |x|^2 P_T^{n-1}(x_0, dx) \leq C_{13} (1 + C_{14} + C_{14}^2 + \cdots + C_{14}^{n-2}) + C_{14}^{n-1} |x_0|^2$$

holds, then by (3.25) we have

$$\begin{aligned} \int_{\mathbb{R}^d} |x|^2 P_T^n(x_0, dx) &= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |x|^2 P_T^{n-1}(x_0, dy) P_T(y, dx) = \int_{\mathbb{R}^d} \mathbb{E}|X_T^y|^2 P_T^{n-1}(x_0, dy) \\ &\leq C_{13} + C_{14} \int_{\mathbb{R}^d} |y|^2 P_T^{n-1}(x_0, dy) \\ &\leq C_{13} + C_{14} [C_{13} (1 + C_{14} + C_{14}^2 + \cdots + C_{14}^{n-2}) + C_{14}^{n-1} |x_0|^2] \\ &= C_{13} (1 + C_{14} + C_{14}^2 + \cdots + C_{14}^{n-1}) + C_{14}^n |x_0|^2. \end{aligned} \quad (3.26)$$

Therefore, we have, for any $n \geq 1$,

$$\int_{\mathbb{R}^d} |x|^2 P_T^n(x_0, dx) \leq \frac{C_{13}}{1 - C_{14}} + |x_0|^2, \quad (3.27)$$

and moreover,

$$\int_{\mathbb{R}^d} |x|^2 \mu_n(dx) = \frac{\sum_{k=1}^n \int_{\mathbb{R}^d} |x|^2 P_T^k(x_0, dx)}{n} \leq \frac{C_{13}}{1 - C_{14}} + |x_0|^2.$$

Using the Chebyshev inequality, we obtain

$$\sup_n \mu_n(|\cdot|^2 > r) \leq \frac{1}{r} \left(\frac{C_{13}}{1 - C_{14}} + |x_0|^2 \right) \rightarrow 0, \quad r \rightarrow \infty,$$

which shows the tightness of $\{\mu_n\}_{n \geq 1}$.

So, by the Prohorov theorem, there exists a probability μ and a subsequence μ_{n_k} such that

$\mu_{n_k} \rightarrow \mu$ weakly as $n_k \rightarrow \infty$. For simplicity of notation, we denote $\mu_n \rightarrow \mu$ weakly as $n \rightarrow \infty$. Now we will prove that μ is invariant for $(P_T^n)_{n \geq 1}$.

Denote $C_b(\mathbb{R}^d)$ by the set of all bounded continuous functions on \mathbb{R}^d .

For any $f \in C_b(\mathbb{R}^d)$, by (3) of Corollary 3.3, we know that $P_T^n f \in C_b(\mathbb{R}^d)$, $\forall n \geq 1$. Moreover, we deduce that, for all $l \in \mathbb{N}$,

$$\begin{aligned} \mu(P_T^l f) &= \lim_{n \rightarrow \infty} \mu_n(P_T^l f) \\ &= \lim_{n \rightarrow \infty} \frac{\sum_{k=1}^n P_T^{k+l} f(x_0)}{n} \\ &= \lim_{n \rightarrow \infty} \frac{\sum_{m=1}^n P_T^m f(x_0)}{n} + \lim_{n \rightarrow \infty} \frac{\sum_{m=n+1}^{n+l} P_T^m f(x_0)}{n} - \lim_{n \rightarrow \infty} \frac{\sum_{m=1}^l P_T^m f(x_0)}{n} \\ &= \mu(f). \end{aligned}$$

The proof is complete.

4 Integration by parts formula

In this part, we will establish the Driver integration by parts formulas for SDEs driven by fractional Brownian motion. As a consequence, the shifted Harnack inequalities are presented. In contrast to known Harnack inequality, in the shifted Harnack inequality, a reference function is shifted rather than the initial point.

Now consider the following SDE with additive fractional noise on \mathbb{R}^d :

$$dX_t = b(t, X_t)dt + dB_t^H, \quad X_0 = x. \quad (4.1)$$

Throughout the section, we will give some regularity assumptions on the coefficients $b(\cdot, \cdot) : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}^d$.

- (i) If $H < \frac{1}{2}$, the mapping $t \rightarrow b(t, 0)$ is bounded on $[0, T]$ and $|\nabla b(t, \cdot)(x)| \leq \bar{K}_1$, $\forall t \in [0, T]$, $x \in \mathbb{R}^d$,
- (ii) If $H > \frac{1}{2}$, b is independent of the time variable, and b is differentiable on space variable, $|\nabla b| \leq K_1$, $|\nabla b(x) - \nabla b(y)| \leq K_2|x - y|$,

where \bar{K}_1 is positive constant and K_1, K_2 are the same as (H1).

By [20] and [21], we know that the equation (4.1) has a unique solution.

Theorem 4.1 *Let $y \in \mathbb{R}^d$ be fixed. For all f satisfying $f, \nabla_y f \in \mathcal{B}_b(\mathbb{R}^d)$, we get,*

- (1) if $H < \frac{1}{2}$, $P_T(\nabla_y f)(x) = \mathbb{E}(f(X_T^x)N_T^1)$, where the random variable N_T^1 is given by

$$N_T^1 = \frac{1}{\Gamma(\frac{1}{2} - H)T} \int_0^T \left\langle s^{H-\frac{1}{2}} \int_0^s \frac{r^{\frac{1}{2}-H}}{(s-r)^{\frac{1}{2}+H}} [y - r \nabla_y b(r, \cdot)(X(r))] dr, dW_s \right\rangle;$$

(2) if $H > \frac{1}{2}$, $P_T(\nabla_y f)(x) = \mathbb{E}(f(X_T^x)N_T^2)$, where the random variable N_T^2 is shown by

$$\begin{aligned} N_T^2 = & \frac{1}{\Gamma(\frac{3}{2}-H)T} \left\{ \int_0^T \left\langle s^{\frac{1}{2}-H} [y - s \nabla_y b(X(s))], dW_s \right\rangle \right. \\ & + (H - \frac{1}{2}) \int_0^T \left\langle s^{H-\frac{1}{2}} \int_0^s \frac{s^{\frac{1}{2}-H} - r^{\frac{1}{2}-H}}{(s-r)^{\frac{1}{2}+H}} [y - r \nabla_y b(X(r))] dr, dW_s \right\rangle \\ & \left. + (H - \frac{1}{2}) \int_0^T \left\langle \int_0^s \frac{r \nabla_y b(X(r)) - s \nabla_y b(X(s))}{(s-r)^{\frac{1}{2}+H}} dr, dW_s \right\rangle \right\}. \end{aligned}$$

Proof Without lost of generality, we show the result for $d = 1$ for simplicity. For fixed initial point x , let X_t solve (4.1). On the other hand, for each $\epsilon \in [0, 1]$, let X_t^ϵ solve the following equation

$$dX_t^\epsilon = b(t, X_t)dt + dB_t^H + \frac{\epsilon}{T}ydt, \quad X_0^\epsilon = x, \quad t \in [0, T]. \quad (4.2)$$

It is clear that $X_t^\epsilon = X_t + \frac{\epsilon}{T}y$, $t \in [0, T]$. In particular, $X_T^\epsilon = X_T + \epsilon y$.

Define $\xi_t := b(t, X_t) - b(t, X_t^\epsilon) + \frac{\epsilon}{T}y$ and $\bar{B}_t^H = B_t^H + \int_0^t \xi_s ds$. Then we may rewrite (4.2) as follows

$$dX_t^\epsilon = b(t, X_t^\epsilon)dt + d\bar{B}_t^H, \quad X_0^\epsilon = x, \quad t \in [0, T]. \quad (4.3)$$

Note that $\int_0^\cdot \xi_r dr \in I_{0+}^{H+\frac{1}{2}}(L^2([0, T]))$.

Indeed, if $H < \frac{1}{2}$, it suffices to show that $\int_0^T \xi_r^2 dr < \infty$, a.s. If $H > \frac{1}{2}$, a similar argument to step 1 of Theorem 3.1 can deduce the desired result.

Therefore, according to the integral representation for fractional Brownian motion, we obtain

$$\bar{B}_t^H = \int_0^t K_H(t, s) d\bar{W}_s,$$

where $\bar{W}_t = W_t + \int_0^t (K_H^{-1} \int_0^\cdot \xi_r dr)(s) ds$.

Now let

$$R_\epsilon := \exp \left[- \int_0^T \left(K_H^{-1} \int_0^\cdot \xi_r dr \right) (s) dW_s - \frac{1}{2} \int_0^T \left(K_H^{-1} \int_0^\cdot \xi_r dr \right)^2 (s) ds \right].$$

Following the argument of [13, Theorem 4.1] ($H < \frac{1}{2}$) and Theorem 3.1 ($H > \frac{1}{2}$), we may conclude that the process \bar{B}^H is a fractional Brownian motion under the probability $R_\epsilon P$, i.e. (X, X^ϵ) is a coupling by change of measure with changed probability $R_\epsilon P$, and moreover the relations

$$\frac{d}{d\epsilon} R_\epsilon|_{\epsilon=0} = -N_T^1, \quad H < \frac{1}{2}; \quad \frac{d}{d\epsilon} R_\epsilon|_{\epsilon=0} = -N_T^2, \quad H > \frac{1}{2},$$

holds in $L^1(P)$.

Hence, due to [34, Theorem 2.1], we get the desired results.

As the Bismut derivative formula implies the Harnack inequality, we also deduce the following shift Harnack inequality from the Driver integration by parts formula.

Corollary 4.2 *Under the assumptions of Theorem 4.1. For any $f \in \mathcal{B}_b(\mathbb{R}^d)$, we have the following result:*

(1) if $H < \frac{1}{2}$, then, for each $y \in \mathbb{R}^d$,

$$(P_T f)^p \leq (P_T \{f(y + \cdot)\}^p) \exp \left[\frac{p}{p-1} \left(\frac{B(\frac{3}{2} - H, \frac{1}{2} - H)}{\Gamma(\frac{1}{2} - H)} \right)^2 \frac{(1 + \bar{K}_1 T)^2}{T^{2H} 2(1 - H)} |y|^2 \right];$$

(2) if $H > \frac{1}{2}$, then, for every $y \in \mathbb{R}^d$ satisfying $|y| \leq \frac{p-1}{p} \sqrt{\frac{\lambda_0}{2A_2}}$,

$$(P_T f)^p \leq (P_T \{f(y + \cdot)\}^p) \exp \left[\frac{p}{p-1} \left(A_1 + \frac{A_2}{\lambda_0} \log B_0 \right) |y|^2 \right].$$

Proof Note that

$$\begin{aligned} \langle N^1 \rangle_T &= \frac{1}{(\Gamma(\frac{1}{2} - H)T)^2} \int_0^T s^{2H-1} \left\{ \int_0^s \frac{r^{\frac{1}{2}-H}}{(s-r)^{\frac{1}{2}+H}} [y - r \nabla_y b(r, \cdot)(X(r))] dr \right\}^2 ds \\ &\leq \left(\frac{1 + \bar{K}_1 T}{\Gamma(\frac{1}{2} - H)T} \right)^2 |y|^2 \int_0^T s^{2H-1} \left[\int_0^s \frac{r^{\frac{1}{2}-H}}{(s-r)^{\frac{1}{2}+H}} dr \right]^2 ds \\ &= \left(\frac{B(\frac{3}{2} - H, \frac{1}{2} - H)}{\Gamma(\frac{1}{2} - H)} \right)^2 \frac{(1 + \bar{K}_1 T)^2}{T^{2H} 2(1 - H)} |y|^2, \end{aligned}$$

and

$$\langle N^2 \rangle_T \leq (A_1 + A_2 \|B^H\|_{H-\delta}^2) |y|^2,$$

where A_1 and A_2 are defined before. By [34, Proposition 2.3] and Theorem 4.1, we get the desired result.

An important application is to show the existence of the density with respect to the Lebesgue measure for solutions of (4.1).

Corollary 4.3 *Under the assumptions of Theorem 4.1. If $H < \frac{1}{2}$, then the law of the random variable X_T^x has a density with respect to the Lebesgue measure on \mathbb{R}^d .*

Proof Let

$$C_{15} = \frac{p}{p-1} \left(\frac{B(\frac{3}{2} - H, \frac{1}{2} - H)}{\Gamma(\frac{1}{2} - H)} \right)^2 \frac{(1 + \bar{K}_1 T)^2}{T^{2H} 2(1 - H)}.$$

By (1) of Corollary 4.2, we get, for any $f \in \mathcal{B}_b(\mathbb{R}^d)$,

$$(P_T f(x))^p e^{-C_{15}|y|^2} \leq (P_T \{f(y + \cdot)\}^p)(x)$$

Taking $f = I_A$, A is Lebesgue-null set on \mathbb{R}^d , and integrating both sides with respect to dy on the above relation, we deduce that

$$(P_T I_A(x))^p \int_{\mathbb{R}^d} e^{-C_{15}|y|^2} dy \leq 0,$$

which implies the desired result.

Remark 4.4 In [34], the author showed that the study of the Driver integration by parts formula and shift Harnack inequality was in general more difficult than that of the Bismut derivative formula and Harnack inequality. Moreover, he gave some applications of integration by parts formula and shift Harnack inequality, for instance, to estimate the density with respect to the Lebesgue measure for distributions and Markov operators.

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